ET4147 Wavefield Imaging Lecture 1 - 30/04/2015

Introduction to Field Problems

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1 Introduction

How, for a set of measurements, an unknown phenomena can be reconstructed? How the conditions (medium) in which this event is observed can be estimated by field measurements?
Wavefield imaging addresses mainly problems like those called *inverse problems*, where from a set of field measurements the sources which produce the fields (inverse source problem) or the propagation media parameters (inverse medium problem) are wanted to be estimated. Sadly this task is not always straightforward. Most of the time inverse problem are **hard** to solved and present tough conditions for finding appropriate approximations to their solutions. However, several methods to solve inverse problems are mature enough for engineers to take advantage of them.
This course focuses mainly in techniques to solve inverse problems in a general framework, without any assumptions of the nature of the field, by using the general wave equation methods to solve both inverse source and inverse medium problems are studied, giving a deep understanding of the difficulty of the task and at the same time enough tools to deal with this kind of problems in a wide range of applications.

2 Field Problems

Before going directly to inverse problems, a digression towards forward problems will be made.
As mentioned before, inverse problems are the ones which from a set of field measurements $\mathbf{U}^{SC}$ the sources $\mathcal{S}$ or the media parameters $\Phi = \{\mu_1, \mu_2, ...\}$ are going to be estimated. In the case of forward problems the media Φ and the $\mathcal{S}$ are known and now the field $\mathbf{U}^{SC}$ is wanted to be computed.

In other words, we can state the forward problem as following:

Forward Problem
Given a source and a known media (configuration) compute the field.

In general, forward problems are **non-linear but they are well-posed**.
For a problem to be well-posed the Hadamard Criteria should be met.

Hadamard Criteria

- Existence
- Uniqueness
- Stability

If one of these properties does not hold, the problem is called ill-posed.

Now that the forward problem is formally stated, we continue our discussion toward inverse problems. In the case of inverse problems, we can find our self trying to estimate the sources which generate the measured field or the configuration of media which modify the field produced by a known source.
Formally these two type of inverse problem are defined as:

Inverse Source Problem
Given field measurements compute the position of the source.

This kind of problems are in general **linear and ill-posed**. They are the cornerstone for inverse medium problem, in which several instances of inverse source problems need to be computed before arriving to a configuration estimate. The upside of inverse source is that they are linear, which allows fairly simple techniques to find good estimates for the source location.

Inverse Medium Problem
Given a known sources and field measurements, compute the configuration of the propagation media.

Unfortunately this type of problems are the hardest ones. They are generally **non-linear and ill-posed**. This means that in order to solve this problems, if a solution exist, computationally complex method should be used and even then there is no guarantee that the solution is stable (small perturbations in the data can blow completely our result).

Now that we have a good idea about what ahead us, it is possible to continue with the basic tool for wavefield imaging: the *wave equation*.

3 Wave Equation

All the theory of wavefield imaging is based in the general wave equation. It is always possible to not relay in any particular type of wave to derive the methods presented in the course, but it is important to explore the two most common type of wave sources in imaging: *electromagnetic and acoustic*.
First, we will survey the basic description of the wave behavior of these phenomena and then the general framework based in the Helmoltz equation will be presented in order to forget the nature of the wave and just focus in the structure for solving general problems.

3.1 Electromagnetic Equations

Now we are in the domain of the well-known Maxwell.

Maxwell Equations

$$-\nabla \times \mathbf{H} + \mathbf{J} + \partial_t \mathbf{D} = -\mathbf{J}^{ext} \quad \text{Maxwell-Ampere} \quad (1)$$
$$\nabla \times \mathbf{E} + \partial_t \mathbf{B} = 0 \quad Faraday's Law \quad (2)$$

Constitutive Relations

$$\mathbf{J} = \sigma \mathbf{E},\ \mathbf{D} = \epsilon \mathbf{E},\ \mathbf{B} = \mu \mathbf{H}$$

where

$$\sigma := \text{Condutivity},\ \epsilon := \text{Permitivity},\ \mu := \text{Permeability}$$

It is seen directly from the Maxwell Equations that they are wave equations (the order of the spatial terms is equal to the time terms).

Now, the *compatibility relations* are derived by using the following relation:

$$\nabla \cdot (\nabla \times \mathbf{A}) = 0$$

Taking the divergence from Eq.(1) and (2) we obtain

$$\nabla \cdot \mathbf{J} + \partial_t \nabla \cdot \mathbf{D} = -\nabla \cdot \mathbf{J}^{ext} \quad (3)$$
$$\partial_t \nabla \cdot \mathbf{B} = 0 \quad (4)$$

by making the following change of variables

$$\mathbf{J}^{tot} = \mathbf{J}^{tot} + \mathbf{J}^{ext},\ \nabla \cdot \mathbf{D} = \rho$$

Eq. (3) can be expressed as

$$\nabla \cdot \mathbf{J}^{tot} + \partial_t \rho = 0 \quad \text{Conservation of charge} \quad (5)$$

Finally recalling (4)

$$\partial_t \nabla \cdot \mathbf{B} = 0$$

this implies that $\nabla \cdot \mathbf{B} = K$, where K is a constant term through time. Considering the initial condition at $t = 0$, it is concluded that

$$\nabla \cdot \mathbf{B} = 0\ \forall\ t \quad \text{Closeness of Magnetic Flow} \quad (6)$$

The Maxwell equations can be then summarize as:

$$\begin{gathered} -\nabla \times \mathbf{H} + \sigma \mathbf{E} + \epsilon \partial_t \mathbf{E} = -\mathbf{J}^{ext} \\ \nabla \times \mathbf{E} + \mu \partial_t \mathbf{H} = 0 \\ \nabla \cdot \mathbf{J}^{tot} + \partial_t \rho = 0 \\ \nabla \cdot \mathbf{B} = 0 \\ \rho = \nabla \cdot \mathbf{D} \end{gathered}$$

In particular some special cases for these equations can be derived when some of the terms are neglected.

- **Electroquasistatics** [$\nabla \times \mathbf{E} = 0$]
  Here a static field approximation is used by neglecting $\mu \partial_t \mathbf{H}$ term. This approximation is extensively used, specially in Electrical Impedance Tomography (EIT).

- **Magnetoquasistatics** [Diffusion Process]
  When the term $\epsilon \partial_t \mathbf{E}$ is neglected a diffusion process is now present as the equations are not the ones from a wave anymore, and they are not decoupled as in the previous case. This type of equations are usually used in oil and gas industry problems.

- **Static Fields** [No time variations]
  Now all time variant terms are set to zero ($\partial_t = 0$), which leads to the series of equations

$$\begin{array}{ll} \nabla \times \mathbf{E} = 0 & \nabla \times \mathbf{H} = \mathbf{J}^{tot} \\ \rho = \nabla \cdot \mathbf{D},\ \mathbf{D} = \epsilon \mathbf{E} & \nabla \cdot \mathbf{B} = 0,\ \mathbf{B} = \mu \mathbf{H} \\ & \nabla \cdot \mathbf{J}^{tot} = 0 \end{array}$$

3.2 Acoustic Waves (In Fluids or Gases)

In this section acoustic waves equation in fluid or gases media is considered. We already be able to see some similarities with the EM case just by making the proper analogies in the acoustic problem.

The acoustic waves equations are the following

$$\nabla p + \rho \partial_t \mathbf{v} = \mathbf{f} \quad \text{Equation of Motion} \quad (1)$$
$$\nabla \mathbf{v} + \kappa \partial_t p = \phi \quad \text{Deformation Equation} \quad (2)$$

where

$$\rho := \text{Density} \qquad \kappa := \text{Elasticity}$$
$$p := \text{Pressure[Pa]} \quad \mathbf{v} := \text{Particle Velocity[m/s]}$$

With the assumption of a homogeneous media and ρ, κ constant, the scalar case for the pressure wave equations can be derived.

In order to eliminate $\mathbf{v}$

$$\nabla \cdot \nabla p + \rho \partial_t \nabla \cdot \mathbf{v} = \nabla \cdot \mathbf{f} \qquad div[Eq.(1)] \quad (3)$$
$$\partial_t(\nabla \cdot \mathbf{v}) + \kappa \partial_{tt} p = \partial_t \phi \qquad \partial_t[Eq.(2)] \quad (4)$$
$$\nabla \cdot \mathbf{f} - \nabla^2 p + \kappa \rho \partial_{tt} p = -\rho \partial_t \phi \quad (4) \to (3) \quad (5)$$

by rearranging (5) the scalar wave equation for the pressure is obtained

$$\nabla^2 p - \kappa \rho \partial_{tt} p = -\rho \partial_t \phi + \nabla \cdot \mathbf{f} \qquad (6)$$

3.3 General Wave Equation

So far the wave equations for the electromagnetic and acoustic case have been derived. In most of the imaging problem the nature of the wave is needed to be known, not to solve the problem, but to completely understand the interaction of the field with the matter.
Under this assumption we will continue the rest of the course, and will derive methods to solve field imaging problems, only focusing in the general wave equation described by

$$\nabla^2 \mathbf{u} - \tfrac{1}{c^2} \partial_{tt} \mathbf{u} = -\mathbf{q} \quad (1)$$

To be able to continue with the derivation of the Helmolts Equation, Laplace and Fourier transform are needed.

Laplace Transform w.r.t time

$$\hat{f}(s) = \int_{t=0}^{\infty} exp(-st) f(t) dt \quad \text{LT}$$

$$f(t) = \tfrac{1}{2i\pi} \int_{s_0 - i\infty}^{s_0 + i\infty} exp(st) \hat{f}(s) ds \quad \text{ILT}$$

To Fourier Transform $\to s = -i\omega$

$$\tilde{f}(\omega) = \int_{t=0}^{\infty} exp(i\omega t) f(t) dt \quad FT$$
$$f(t) = \tfrac{1}{2\pi} \int_{\omega=0}^{\infty} exp(-i\omega t) \tilde{f}(\omega) d\omega \quad IFT$$

By making the following assumptions (scalar case)

$$u(\mathbf{x}, t = 0) = 0, \ \partial_t u(\mathbf{x}, t = 0) = 0 \ \forall \mathbf{x} \in \mathbb{R}^3$$

the following representation of the wave equation are obtained

Laplace Domain

Modified Helmholtz Equation
$$\nabla^2 \hat{u} - \hat{\gamma}^2 \hat{u} = -\hat{q}$$
$$\hat{\gamma} = \tfrac{s}{c}$$

Fourier Domain

Helmoltz Equation
$$\nabla^2 \tilde{u} + k^2 \tilde{u} = -\tilde{q}$$
$$k = \tfrac{\omega}{c} \text{ (wave number)}$$

References

[1] B. Saleh, "Introduction to Subsurface Imaging", *Cambridge Press,* 2011.

ET4147 Wavefield Imaging	Lecture 2 - 07/05/2015
Forward and Inverse Source Problems	
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1 Introduction

In this lecture the forward problem is further explored. By means of the Green function the field is shown to be reconstructed when the configuration (medium properties) and source are known. Furthermore the inverse source problem is addressed in both near and far field set ups. The far field problem, the field measurements are acquired very far from the emitting source, is simplified by means of the Fresnel approximation, which leads a natural Fourier representation of the solution. For the near field case the solution is not as straightforward. However, by proper selection of source kernels (a prior knowledge), the source can be localized, if possible, by means of linear algebra (SVD decomposition).

2 Forward Problem

As discussed in the previous lecture, forward problems involves the computation of the field when the configuration and sources are given. In this lecture, the solution for this problem will be derived for a homogeneous media (Φ constant over all space).

Recalling the modified scalar wave equation in the $s-$domain for $\hat{q} \neq 0$

$$\nabla^2 \hat{u} - \hat{\gamma}^2 u = -\hat{q},\ \hat{\gamma} = \frac{s}{c}$$

Defining the operator $\mathcal{D}$ as

$$\mathcal{D} = \nabla^2 - \hat{\gamma}^2$$

the modified wave equation can be succinctly write as

$$\mathcal{D}\hat{u} = -\hat{q}$$

Now, the spatial Fourier transform (SFT) is defined by

$$\tilde{f}(\mathbf{k}, s) = \int_{\mathbf{x}\in\mathbb{R}^3} exp(-i\mathbf{k}\cdot\mathbf{x})\hat{f}(\mathbf{x}, s)dV$$

and its inverse (ISFT)

$$\hat{f}(\mathbf{x}, s) = \frac{1}{(2\pi)^3}\int_{\mathbf{k}\in\mathbb{R}^3} exp(i\mathbf{k}\cdot\mathbf{x})\tilde{f}(\mathbf{k}, s)dV$$

note the difference in the definition of SFT with respect the time Fourier Transform (TFT) where the argument in the exponential was set to $i\omega t$.

Making the substitution $s = -i\omega$ and taking the ITFT

$$f(\mathbf{x}, t) = \frac{1}{(2\pi)^4}\int_{\omega=\infty}^{\infty}\int_{\mathbf{k}\in\mathbb{R}^3} exp[-i(\omega t - \mathbf{k}\cdot\mathbf{x})]\tilde{f}(\mathbf{k}, \omega)dVd\omega$$

where it can be seen that the exponential term in the integral represents an infinite sum of planar waves.

By leveraging the relation of ∇ operation in $\mathbf{x}$-domain and $\mathbf{k}$-domain, we can write

$$\nabla^2 = \nabla \cdot \nabla \rightarrow \mathbf{k} \cdot \mathbf{k} = -k^2$$

with $\nabla \rightarrow i\mathbf{k}$.

This will allows us to rewrite our operator $\mathcal{D}$ in the Fourier domain as

$$\tilde{\mathcal{D}} = (k^2 + \tilde{\gamma}^2)$$

Finally the forward problem is easily solved (analytically) by using the inverse operator $\tilde{G} := \tilde{\mathcal{D}}^{-1}$ and applying it to $\tilde{q}$.
The solution for the field $\tilde{u}$ (in the spatial frequency domain) is given by

$$\tilde{u} = \tilde{G}\tilde{q}$$

where

$$\tilde{G} = \frac{1}{k^2 + \tilde{\gamma}^2}$$

which is commonly known as the **Green Function**.

In order to go back to the $\mathbf{x}$-domain the ISFT is computed for $\tilde{G}$

$$\hat{G}(\mathbf{x}, s) = \frac{1}{(2\pi)^3} \int_{\mathbf{k} \in \mathbb{R}^3} \frac{exp(i\mathbf{k} \cdot \mathbf{x})}{k^2 + \hat{\gamma}^2} dV$$

which leads to

$$\hat{G}(\mathbf{x}, s) = \frac{exp(-\hat{\gamma}|\mathbf{x}|)}{4\pi|\mathbf{x}|}, \ |\mathbf{x}| \neq 0$$

Now $\hat{u}$ can be computed by applying a convolution in the $\mathbf{x}$-domain

$$\tilde{u} = \tilde{G}\tilde{q} \leftrightarrow \hat{u}(\mathbf{x}, s) = \int_{\mathbf{x}' \in D_{src}} \hat{G}(\mathbf{x} - \mathbf{x}', s)\hat{q}(\mathbf{x}', s) dV$$

by substituting the Green function in the previous equation

$$\hat{u}(\mathbf{x}, s) = \int_{\mathbf{x}' \in D_{src}} \frac{exp(-s\frac{|\mathbf{x}-\mathbf{x}'|}{c})}{4\pi|\mathbf{x} - \mathbf{x}'|} \hat{q}(\mathbf{x}, s)$$

using the time shift property

$$f(t - \tau) = exp(-s\tau)\hat{f}(s)$$

the expression for the field $u(\mathbf{x}, t)$ can be written as

$$u(\mathbf{x}, t) = \int_{\mathbf{x}' \in D_{src}} \frac{q(\mathbf{x}', t - \frac{|\mathbf{x}-\mathbf{x}'|}{c})}{4\pi|\mathbf{x} - \mathbf{x}'|} dV, \ t \geq 0$$

3 Far Field Approximation

In some applications the field $u(\mathbf{x}, t)$ is desired to be known very far from the source which produces it i.e radioastronmy applications where stars are the field sources. By taking advantages of this particular case ($|\mathbf{x}| \rightarrow \infty$) an approximation known as **Fresnel approximation** can be used in order to simplify the relation between the source and the resulting field.

Performing the change of variable $s = -i\omega$ in the convolution integral

$$\hat{u}(\mathbf{x}, \omega) = \int_{\mathbf{x}' \in D_{src}} \frac{exp(ik|\mathbf{x} - \mathbf{x}'|)}{4\pi|\mathbf{x} - \mathbf{x}'|} \hat{q}(\mathbf{x}', \omega) dV$$

with

$$\hat{\gamma} = \frac{s}{c} = -ik, \; k = \frac{\omega}{c}$$

Now using the fact that $|\mathbf{x}| \to \infty$ and expanding the distance term $|\mathbf{x} - \mathbf{x}'|$

$$|\mathbf{x} - \mathbf{x}'| = [|\mathbf{x}|^2 - 2\mathbf{x} \cdot \mathbf{x}' + |\mathbf{x}'|^2]^{\frac{1}{2}}$$

$$|\mathbf{x} - \mathbf{x}'| = |\mathbf{x}|[1 - \frac{2\mathbf{x} \cdot \mathbf{x}'}{|\mathbf{x}|^2} + (\frac{|\mathbf{x}'|}{\mathbf{x}})^2]^{\frac{1}{2}}$$

as the quadratic term decays faster, the distance term in the exponential can be approximated as

$$|\mathbf{x} - \mathbf{x}'| \approx |\mathbf{x}| - \hat{\mathbf{n}} \cdot \mathbf{x}'$$

with $\hat{\mathbf{n}} = \frac{\mathbf{x}}{|\mathbf{x}|}$.

Finally by using this approximation in the exponential term and substituting the denominator by $|\mathbf{x}|$ a simplified version of the for the field in frequency domain is obtained

$$\hat{u}(\mathbf{x}, \omega) = \int_{\mathbf{x}' \in \mathbb{R}^3} \frac{exp(ik|\mathbf{x}|)}{4\pi|\mathbf{x}|} exp(-ik\hat{\mathbf{n}} \cdot \mathbf{x}') \hat{q}(\mathbf{x}', \omega) dV$$

which is readily seen to be the SFT of the source

$$\hat{u}(\mathbf{x}, \omega) = \frac{exp(ik|\mathbf{x}|)}{4\pi|\mathbf{x}|} \tilde{q}(\mathbf{k}, \omega), \; \mathbf{k} = k\hat{\mathbf{n}} = \frac{\omega}{c}\hat{\mathbf{n}}$$

This last expression gives not only the field but also a useful insight for the measuring processes. When the source is far away, the measurements of field in a given location represent the frequency spectra of the source. This means that far away from our source we measure the spectral distribution of the it, where the term $\tilde{q}(\mathbf{k}, \omega)$ is called the **radiation pattern** of the source and the other term is the already known Green function.

Inverse Source Problem for Far Field
Using this result the inverse source problem is now known. From a the set of field measurements $U^{SC}(\mathbf{x}, \omega)$ far from the source, the product of the inverse of the Green function $\hat{G}^{-1}$ will give the radiation pattern of the source (in a noise free case). By computing the ITFT the source can be found.

4 Near Field Problem

In this kind of problems the field is measured/unknown at the vicinity of the source. For this case an approximation as Frenel's is not possible anymore. For the forward problem this does not represent a problem, solving the convolution integral the field can be computed without issues. However, for inverse source problem a prior knowledge should be used in order to give a structure to the source, which allows its reconstruction.

Focusing our attention to the inverse problem we considered $\hat{q} \neq 0$ unknown. A set of field measurements are taken in locations $\mathbf{x}_k^R, k = 1, 2, ..., m$ near the source. Now the convolution integral for the field is

$$\hat{u}(\mathbf{x}_k^R, \omega) = \int_{\mathbf{x}' \in D_{src}} \hat{G}(\mathbf{x}_k^R - \mathbf{x}', \omega) \hat{q}(\mathbf{x}', \omega) dV, \; k = 1, 2, ..., m$$

by including some a priori knowledge the source $\hat{q}$ is structure by

$$\hat{q} = \sum_{p=1}^{n} q_p \varphi_p(\mathbf{x})$$

where φ_p is known. Using the given structure of the source the integral is rewritten as

$$\hat{u}(\mathbf{x}_k^R, \omega) = \sum_{p=1}^{n} q_p \int_{\mathbf{x}' \in D_{src}} \hat{G}(\mathbf{x}_k^R - \mathbf{x}', \omega) \varphi_p(\mathbf{x}') dV$$

The integral terms are completely known for all p and k. This terms g_{kp} are called **data kernels**.

Stacking all the field measurements in a vector $\mathbf{d} \in C^m$ and the data kernels in a matrix $\mathbf{G} \in C^{m \times n}$, the **data equation** is expressed as

$$\mathbf{d} = \mathbf{Gq}$$

with $\mathbf{q} = [q_1, ..., q_n]^T$.

This compact way to express the problem give a huge advantage. Now the problem is expressed in term of matrices and vectors, allowing a linear algebra kind of approach for its solution. In addition, every new field measurement will add a new row to the data equation which means that if enough measurements are retrieved ($m >> n$), the vector $\mathbf{q}$ will be able to be estimated.

5 SVD-Based Method for Inverse Source Problem

Observing our data equation

$$\mathbf{d} = \mathbf{Gq}$$

we realized that perfect reconstruction of $\mathbf{q}$ is only possible for cases in which the measurements $\mathbf{d}$ are free from noise. In reality, this is far for true. Most of the cases our field measurements will be plagued with different kinds of noise (additive, multiplicative, etc). In order to be able to reconstruct $\mathbf{q}$ through this measurements an error measure should be minimized

In most of the application a proper error measure is not known, an usually a general one is preferred. Least Square (LS) solutions is a widely used approach to tackle this kind of problems.

Least Squares Problem

$$\underset{\mathbf{q}}{\text{minimize}} \quad ||\mathbf{d} - \mathbf{Gq}||_2^2$$

The solution given by LS has the next properties

- Satisfy the normal equations: $\mathbf{G}^H\mathbf{Gq} = \mathbf{G}^H\mathbf{d}$
- For $\mathbf{q}$ being a LS solution, then $\mathbf{q} + \alpha\mathbf{z}$ is also a solution iff $\mathbf{z} \in \mathcal{N}(\mathbf{G})$

This last property can be interpreted as a solution which contains a non radiating source

$$\mathbf{Gq}_{NR} = 0, \ \mathbf{q}_{NR} \neq 0$$

Now that all the components for a LS approach are described, Singular Value Decomposition (SVD) is introduced to give a intuitive way to find the solution to the inverse problem.

Singular Value Decomposition

$$\mathbf{G} = \mathbf{U}\Sigma\mathbf{V}^H$$
$$\begin{array}{ll} \mathbf{U} & \text{Unitary m x m} \\ \mathbf{V} & \text{Unitary n x n} \end{array}$$

$$\Sigma = \begin{bmatrix} \Sigma_r & 0 \\ 0 & 0 \end{bmatrix} \in C^{m \times n}$$

Properties of the SVD

- $\mathbf{U} \rightarrow$ left singular vectors $(range(\mathbf{G}))$
- $\mathbf{V} \rightarrow$ right singular vectors $(range(\mathbf{G}^H)$

Solution of LS using SVD

$$\mathbf{b} := \mathbf{V}^H\mathbf{q} = \begin{bmatrix} \mathbf{b}_1 \\ \mathbf{b}_2 \end{bmatrix} \qquad \mathbf{c} := \mathbf{U}^H\mathbf{d} = \begin{bmatrix} \mathbf{c}_1 \\ \mathbf{c}_2 \end{bmatrix}$$

$$\begin{array}{c} \|\mathbf{d} - \mathbf{G}\mathbf{q}\|_2 = \|\mathbf{U}^H(\mathbf{d} - \mathbf{G}\mathbf{V}\mathbf{V}^H\mathbf{q})\|_2 \\ \|\mathbf{c} - \mathbf{U}^H\mathbf{G}\mathbf{V}\mathbf{b}\|_2 \\ \|\mathbf{c} - \Sigma\mathbf{b}\|_2 \\ \| \begin{bmatrix} \mathbf{c}_1 - \Sigma_r\mathbf{b}_1 \\ \mathbf{c}_2 \end{bmatrix} \|_2 \end{array}$$

$$\therefore \mathbf{b}_1^* = \Sigma_r^{-1}\mathbf{c}_1 \qquad \therefore \|\mathbf{r}^*\|_2 = \|\mathbf{c}\|_2$$

this leads to the solution

$$\mathbf{q} = \mathbf{V}\mathbf{b}^* = \mathbf{V}\begin{bmatrix} \Sigma_r^{-1}\mathbf{c}_1 \\ \mathbf{b}_2 \end{bmatrix}$$

Finally, the solution with the minimum norm (minimum energy) will the one with component $\mathbf{b}_2 = 0$.

References

[1] B. Saleh, “Introduction to Subsurface Imaging”, *Cambridge Press,* 2011.

[2] M. Oristaglio and H. Blok, “Wavefield Imaging and Inversion in Electromagnetisc and Acoustics”, Lectures Notes *TU Delft,* 1995

ET4147 Wavefield Imaging	Lecture 3 - 21/05/2015
Underdetermined Systems and Inverse Medium Problem	
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1 Introduction

In this lecture solutions for underdetermined systems are covered. Alternatives to the costly closed form solutions, given by inversion of an matrix outer product, are also presented and its applications in imaging discussed. In addition, methods involving the SVD using weights are introduced in order to deal with problems with ill-conditioned matrices. Finally, the inverse medium problem is described and it solution in integral form is sketched. Next lecture will cover in detail how the integral form problem is solved by means of Born approximation.

2 LS Solution for Linear Systems

In general, a LS problem from a linear system

$$\mathbf{d} = \mathbf{G}\mathbf{q},\ \mathbf{G} \in C^{m\times n}$$

has a solution of the form

$$\hat{\mathbf{q}} = \mathbf{G}^{\dagger}\mathbf{d}$$

where $\mathbf{G}^{\dagger}$ represents the Moore - Penrose pseudoinverse.

In particular three cases exist for the pseudoinverse depending of the rank of $\mathbf{G}$:

- Common matrix inverse $[m = n]:\ \mathbf{G}^{\dagger} = \mathbf{G}^{-1}$
- Overdetermined System $[m > n]:\ \mathbf{G}^{\dagger} = (\mathbf{G}^H\mathbf{G})^{-1}\mathbf{G}^H$
- Underdetermined System $[m < n]:\ \mathbf{G}^{\dagger} = \mathbf{G}^H(\mathbf{G}\mathbf{G}^H)^{-1}$

If we pay attention to the previous expression, in the case of underdetermined systems the matrix that should be inverted is the outer product of $\mathbf{G}$, which has a size of $m \times m$. Recalling the complexity to invert a matrix ($\mathcal{O}(m^3)$), is easily see that for very large m (as in subsurface exploration) this operation becomes extremely costly. In practice, for cases when m is very large an approximation for the optimal solution is provided by

$$\mathbf{q}^* \approx \mathbf{G}^H\mathbf{d}$$

The usage of this approximation is what it is called **imaging**.

When imaging is used the scaling factor giving by the prohibiting inverse is omitted and only the propagation towards the source (described by $\mathbf{G}^H$) is used. This means that while $\mathbf{G}$ describes the propagation of the field from the source, its hermitian describes the inverse process.

3 Solutions for Underdetermined Systems

Now that we have a flavor of how to solve a undertermined system of linear equations, a common problem in inverse problems is addressed.

From the pseudoinverse definition we have

$$\mathbf{G}^{\dagger} = \hat{\mathbf{V}}\hat{\Sigma}_r^{-1}\hat{\mathbf{U}}^H$$

which is the inverse of the economy size SVD of $\mathbf{G}$.

If the minimum norm solution is given by

$$\mathbf{q}_{mn} = \mathbf{G}^{\dagger}\mathbf{d}$$

it is possible to represent the solution vector by

$$\mathbf{q}_{mn} = \sum_{k=1}^{r} \frac{\mathbf{u}_k^H \mathbf{d}}{\sigma_k}\mathbf{v}_k$$

In real situations the economy size SVD is a truncated version of the original SVD, which retains the first r highest singular values. Unfortunately identifying the point in which the SVD should be truncated is usually not an easy task (if not prior information about the true rank is available). The existence of small singular values affect heavily our solution, making it unstable under noisy conditions. This can be seen from the previous expression for $\mathbf{q}_{mn}$ where the singular values are in denominator of each term.

Most of the *techniques* used to deal with this kind of situations are rather heuristic. In general, a weight factor is included to the previous expression in order to avoid division by small numbers.

$$\hat{\mathbf{q}}_{mn} = \sum_{k=1}^{r} \frac{w_k}{\sigma_k}(\mathbf{u}_k^H \mathbf{d})\mathbf{v}_k$$

A common one called **Thresholded SVD** (TSVD) assign a weight $w_k = 1$ if $k \leq k_o$ and $w_k = 0$ otherwise. Another method, little more elaborated, uses a window function to assign weights. Particularly we can think of the **Triangular Window** as

$$w_k \begin{cases} 1 - \frac{k-1}{k_0} & k \leq k_0 \\ 0 & \text{otherwise} \end{cases}$$

Continuing with the weights approach, we can find more elaborated ideas which are related with the idea of *regularization*. For example a very general method to select the weights is given by

$$w_k = \frac{\sigma_k^2}{\sigma_k^2 + \lambda},\ \lambda > 0$$

A nice interpretation for this chose of weights is borrowing the notion of regularization for branch of science as machine learning or statistical learning. In a nutshell, regularization is the addition of a penalty term weighted by a certain value which represents the importance of the this particular cost. In this particular case the regularized cost function is given by

$$J(\mathbf{q}, \lambda) = \|\mathbf{d} - \mathbf{G}\mathbf{q}\|_2^2 + \lambda\|\mathbf{q}\|_2^2$$

This cost function includes the energy term of the source given by the norm two of the solution vector. As the weight lambda increases, the solution will become smaller and smaller in the norm two sense (smaller energy).

So far we are happy with our general method to get rid of the problematic singular values, but know the question of how to choose λ arise. Sadly, the tuning of λ is completely problem dependent and there is no unique way to do it. In machine learning and statistics cross - validation is widely used to tune λ. In imaging most of the cases the **L Curve** is used as a graphic way to chose a proper value for the parameter. To compute the L Curve a sweep for different values is performed and the result of the cost are plotted in log - log scale. The value of λ which gives the *knee* of the graph is usually considered the best value for the parameter. In addition to the L Curve method, other ways to pick a proper λ can be found in the literature (i.e Morozov Discrepancy).

Finally, it is worthy to say that other types of regularization are possible and they depend of the type of solution that we are looking for. i.e Sparse solution for $\mathbf{q}$

$$J(\mathbf{q}, \lambda) = \|\mathbf{d} - \mathbf{G}\mathbf{q}\|_2^2 + \lambda \|\mathbf{q}\|_1$$

where the norm one is a good heuristic which enforces our desire of a sparse solution vector.

4 Inverse Media Problem / Inverse Scattering

Now that we have the tools to solve an underdetermined system, as is the case of most imaging set ups, the description of the inverse scattering problem is presented.

In the inverse media problem a set of **known** and **controlled** sources are used to generate a field. A set of receiver are deployed in a domain $\mathcal{D}_{rcvr}$ and a unknown object with unknown configuration is going to be imaged through its scattered field. In this set up the velocity of the wave is known in the background media, which is assumed homogeneous.

By following the previous problem description it is possible to identify the source of non-linearity of our problem: Both the field inside the object and its configuration are **unknown**.

In the following part we will state the integral form of our problem to be able to solve it by using the previously discussed techniques when its discrete form is considered (next lecture).

Lets define the equations for the fields as

Incident Field (No Object)

$$\begin{array}{lll} (1) & \nabla^2 u^b + k_b^2 u^b = -q(\mathbf{x}) & \mathbf{x} \notin \mathcal{D}_{obj} \\ (2) & \nabla^2 u^b + k_b^2 u^b = 0 & \mathbf{x} \in \mathcal{D}_{obj}, \ k_b = \frac{\omega}{c_b} \end{array}$$

Total Field

$$\begin{array}{lll} (3) & \nabla^2 u + k_b^2 u = -q & \mathbf{x} \notin \mathcal{D}_{obj} \\ (4) & \nabla^2 u + k^2 u = 0 & \mathbf{x} \in \mathcal{D}_o bj, \ k = \frac{\omega}{c} \end{array}$$

Modifying Eq.(4)

$$\begin{array}{lll} (4^*) & \nabla^2 u + k_b^2 u + (k^2 - k_b^2) u = 0 & \mathbf{x} \in \mathcal{D}_{obj} \\ & \nabla^2 + k_b^2 u = -(k^2 - k_b^2) u & \mathbf{x} \in \mathcal{D}_{obj} \end{array}$$

Now, subtracting (1) from (3) and (2) from (4*)

$$\begin{array}{lll} (5) & \nabla^2 (u - u^b) + k_b^2 (u - u^b) = 0 & \mathbf{x} \notin \mathcal{D}_{obj} \\ (6) & \nabla^2 (u - u^b) + k_b^2 (u - u^b) = -(k^2 - k_b^2) u & \mathbf{x} \in \mathcal{D}_{obj} \end{array}$$

Finally, defining $u^{sc} := u - u^b$

$$\nabla^2 u^{sc} + k_b^2 u^{sc} = -q^{sc}$$

with

$$q^{sc} = \begin{cases} (k^2 - k_b^2)u & \mathbf{x} \in \mathcal{D}_{obj} \\ 0 & \mathbf{x} \notin \mathcal{D}_{obj} \end{cases}$$

Now we can use our knowledge for the source problem as write the relation for the scattering field as

$$u^{sc}(\mathbf{x}, \omega) = \int_{\mathbf{x}' \in \mathcal{D}_{obj}} G(\mathbf{x} - \mathbf{x}') q^{sc}(\mathbf{x}', \omega) dV$$

$$u^{sc}(\mathbf{x}, \omega) = \omega^2 \int_{\mathbf{x}' \in \mathcal{D}_{obj}} G(\mathbf{x} - \mathbf{x}') [\frac{1}{c^2(\mathbf{x})} - \frac{1}{c_b^2}] u(\mathbf{x}', \omega) dV$$

where the term containing the speeds is called the **contrast function** $\delta_m(\mathbf{x})$.

Finally, the integral form for the scattered field is given by

$$u^{sc}(\mathbf{x}, \omega) = \omega^2 \int_{\mathbf{x}' \in \mathcal{D}_{obj}} G(\mathbf{x} - \mathbf{x}') \delta_m(\mathbf{x}) u(\mathbf{x}', \omega) dV$$

References

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[2] M. Oristaglio and H. Blok, "Wavefield Imaging and Inversion in Electromagnetisc and Acoustics", Lectures Notes *TU Delft,* 1995