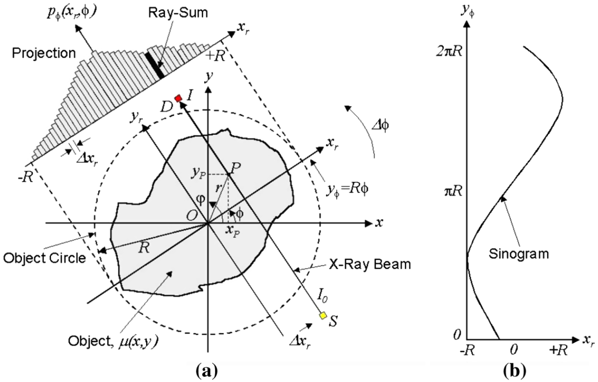
# SDR Applications Outline

Applications and Examples

- CT Reconstruction
- Sensor Network Localization

# CT Reconstruction

## Computed Tomography Problem




GammaScorpion: mobile gamma-ray tomography system for early detection of basal stem rot in oil palm plantations. J. Abdullah, et al 2013

# CT Reconstruction

"Natural" Approach for Inversion

$$g(x,y) = \frac{1}{2\pi}\mathcal{B}\{\mathcal{H}\{\frac{\partial}{\partial\rho}\mathcal{R}\{g(x,y)\}\}\}$$

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where

$$\mathcal{R}f(L) = \int_L f(\mathbf{x})|d\mathbf{x}|$$

# CT Reconstruction

A MIMO interpretation of the CT continuous problem...

$$\breve{g}(\rho, \theta) = \mathcal{R}\{g(x, y)\}$$

Assuming fixed voxels (discretization) and known sampling (relaxing resolution) a linear system can be used

$$\mathbf{b} = \mathbf{A}\mathbf{x}$$

Insights?

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**A** is **HUGE**

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Insights?
**A** is **HUGE** but **Sparse**.

That is why common approaches use *iterative sparse sensitive methods* [1]

[1] Angenaeherte Aufloesung von Systemen linearer Gleichungen. S. Kaczmarz, 1937

# CT Reconstruction

Reconstruction as an Optimization Problem (Typical MIMO)

$$\underset{x \in \mathcal{I}}{\text{minimize}} \|\mathbf{b} - \mathbf{A}\mathbf{x}\|^2$$

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Homogeneous *QCQP* representation

$$\underset{x \in \mathcal{I}, t \in \Re}{\text{minimize}} \begin{bmatrix} \mathbf{x}^T & t \end{bmatrix} \begin{bmatrix} \mathbf{A}^T\mathbf{A} & -\mathbf{A}^T\mathbf{b} \\ -\mathbf{b}^T\mathbf{A} & \|\mathbf{b}\|^2 \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ t \end{bmatrix}$$
$$\text{s.t} \quad t^2 = 1$$

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SDR Re-Statement

$$\begin{aligned} \underset{\mathbf{X}}{\text{minimize}} \, & Tr(\mathbf{C}\mathbf{X}) \\ \text{s.t} \quad \mathbf{X} \succeq 0, \; & \mathbf{X}(n+1, n+1) = 1 \end{aligned}$$

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Ideas about how the solution would be?

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Ideas about how the solution would be? Solution is **Low Rank!** [2]

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[2] Remember the rank is upperbounded by ...

# Sensor Network Localization

In other words...

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**Feasibility Problem**

$$\text{Find } \; x_j \in \Re^{dim}, \; j = 1, \ldots, n$$
$$\|x_j - x_i\|^2 = d_{ji}^2, \; \forall (j,i) \in N_x$$
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Or its SDP Formulation

$$\begin{array}{ll} \text{Find} & \mathbf{X} \in \Re^{dim \times n}, \; \mathbf{Y} \in \Re^{n \times n} \\ \text{s.t} & (e_i - e_j)^T \mathbf{Y} (e_i - e_j) = d_{ji}^2, \; \forall (j,i) \in N_x \\ & \begin{pmatrix} a_k \\ -e_j \end{pmatrix}^T \begin{pmatrix} \mathbf{I}_{dim} & \mathbf{X} \\ \mathbf{X}^T & \mathbf{Y} \end{pmatrix} \begin{pmatrix} a_k \\ -e_j \end{pmatrix} = d_{jk}, \forall (j,k) \in N_a \\ & \mathbf{Y} = \mathbf{X}^T \mathbf{X} \end{array}$$

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What *should* be relaxed here?

# Sensor Network Localization

Equivalent *Rank* Constrained SDP

$$
\begin{aligned}
\text{find} \quad & \mathbf{Z} \\
\text{s.t} \quad & Tr(\mathbf{M}_{ji}\mathbf{Z}) = d_{ji}^2, \forall (j,i) \in N_x \\
& Tr(\mathbf{M}_{jk}\mathbf{Z}) = d_{jk}^2, \forall (j,k) \in N_a \\
& \mathbf{Z}_{1:dim,1:dim} = \mathbf{I}_{dim},\ \mathbf{Z} \succeq 0,\ rank(\mathbf{Z}) = dim
\end{aligned}
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Equivalent *Rank* Constrained SDP

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Under what conditions will it be a solution to the original problem (SNL)?

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Under what conditions will it be a solution to the original problem (SNL)?

Uniquely Localizable + Free from Noise [3]

[3] Theory of semidefinite programming for sensor network localization,A.So and Y. Ye, 2007.

# Sensor Network Localization

How to apply SDR under noise?

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How to apply SDR under noise?

Nothing changes, the solution still will be unique (if feasible).

**Common Approaches**

- Thumb Rule (Uncertainty Interval)
- Maximum Likelihood Estimator

$$\begin{array}{ll} \text{minimize} & \sum_{(j,i)\in N_x} \frac{1}{\sigma_{ji}}\epsilon_{ji} + \sum_{(j,k)\in N_a} \frac{1}{\sigma_{jk}}\epsilon_{jk} \\ \text{s.t} & (\|x_j - x_i\| - d_{ji})^2 = \epsilon_{ji},\ \forall (j,i) \in N_x \\ & (\|x_j - x_k\| - d_{jk})^2 = \epsilon_{jk},\ \forall (j,k) \in N_a \end{array}$$